THE INDUCED MINKOWSKI ORDERED WEIGHTED AVERAGING DISTANCE OPERATOR

José M. Merigó 1 Montserrat Casanovas 1

1 Department of Business Administration, University of Barcelona, Av. Diagonal 690, 08034 Barcelona, Spain, {jmerigo, mcasanovas}@ub.edu

Abstract

The Minkowski distance is a distance measure that generalizes a wide range of other distances such as the Hamming and the Euclidean distance. In this paper, we develop a generalization of the Minkowski distance by using the induced ordered weighted averaging (IOWA) operator. We will call it the induced Minkowski OWA distance (IMOWAD). Then, we are able to obtain a wider range of distance measures that includes the Minkowski distance, the Minkowski OWA distance (MOWAD), the induced OWAD (IOWAD), etc. We also present a further generalization that uses quasi-arithmetic means. We will call it the Quasi-IOWAD operator. We end the paper with a numerical example of the new approach.

Keywords: Minkowski distance, Aggregation operators, IOWA operator, Decision making.

1 INTRODUCTION

The distance measures are very useful techniques that have been used in a wide range of applications such as fuzzy set theory, decision making, operational research, etc. The Minkowski distance is one of the main distance measures because it generalizes a wide range of other distances such as the Hamming distance, the Euclidean distance, etc. Often when calculating distances, we want an average result of all the individual distances. We call this the normalization process. In the literature, we find mainly three types of normalized distances. The first one is when we use the arithmetic mean and it is known as the normalized Minkowski distance (NMD). The second one is when we use the weighted average (WA) and it is known as the weighted Minkowski distance (WMD). The third one is when we use the ordered weighted averaging (OWA) operator [1-17] and it is known as the Minkowski ordered weighted averaging distance (MOWAD) operator [5,9]. Note that the MOWAD includes the NMD and the WMD as special cases.

Sometimes, when normalizing the Minkowski distance with the OWA operator, it would be interesting to consider a more general formulation of the attitudinal character. A very useful technique for doing this is the induced ordered weighted averaging (IOWA) operator [14,16]. The IOWA operator provides a parameterized family of aggregation operators such as the maximum, the minimum, the average and the OWA operator.

In this paper, we suggest a new type of distance measure consisting in normalizing the Minkowski distance by using the IOWA operator. Then, the normalization developed will be able to reflect complex attitudinal characters. We will call this generalization as the induced Minkowski OWA distance (IMOWAD) operator. The main advantage of this operator is that it generalizes a wide range of distances such as the NMD, the WMD, the MOWAD [5,9], the induced OWA distance (IOWAD) [7], the induced Euclidean OWA distance (IEOWAD) [8] and a lot of other particular cases. Then, by using this generalization, we get a more complete view of the aggregation process of the individual distances.

We further generalize the IMOWAD operator by using quasi-arithmetic means. As a result, we get the Quasi-IOWAD operator. We also develop an application of the new approach in a decision making problem about selection of investments. We will see that depending on the particular type of IMOWAD operator used, the results may lead to different decisions.

This paper is organized as follows. In Section 2, we briefly review some basic concepts about the Minkowski distance and the IOWA operator. In Section 3 we present the IMOWAD operator. Section 4 analyzes different families of IMOWAD operators. In Section 5 we present the Quasi-IOWAD operator. Section 6 develops a numerical example of the new generalization. Finally, in Section 7 we summarize the main conclusions.
2 PRELIMINARIES

2.1. NORMALIZED MINKOWSKI DISTANCE

The normalized Minkowski distance is a distance measure that generalizes a wide range of distances such as the normalized Hamming distance and the normalized Euclidean distance. In fuzzy set theory, it can be useful, for example, for the calculation of distances between fuzzy sets, interval-valued fuzzy sets, intuitionistic fuzzy sets, etc. It can be formulated for two sets A and B as follows.

Definition 1. A normalized Minkowski distance of dimension n is a mapping \(d_{\nu}: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}\) such that:

\[
d_{\nu}(A, B) = \left( \frac{1}{n} \sum_{i=1}^{n} |a_i - b_i|^\lambda \right)^{1/\lambda}
\]  

where \(a_i\) and \(b_i\) are the \(i\)th arguments of the sets A and B and \(\lambda\) is a parameter such that \(\lambda \in (-\infty, \infty)\).

If we give different values to the parameter \(\lambda\), we can obtain a wide range of special cases. For example, if \(\lambda = 1\), we obtain the normalized Hamming distance (NHD). If \(\lambda = 2\), the normalized Euclidean distance (NED).

Sometimes, when normalizing the Minkowski distance, we prefer to give different weights to each individual distance. Then, the distance is known as the weighted Minkowski distance. For two sets A and B, it can be defined as follows.

Definition 2. A weighted Minkowski distance of dimension n is a mapping \(d_{\nu w}: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}\) that has an associated weighting vector \(W\) of dimension n such that:

\[
d_{\nu w}(A, B) = \left( \sum_{i=1}^{n} w_i |a_i - b_i|^\lambda \right)^{1/\lambda}
\]

where \(w_i\) are the weights associated with the \(i\)th argument of the sets A and B and \(\lambda\) is a parameter such that \(\lambda \in (-\infty, \infty)\).

In this case, we can also obtain a wide range of special cases by using different values in the parameter \(\lambda\). For example, if \(\lambda = 1\), we obtain the weighted Hamming distance (WHD). If \(\lambda = 2\), the weighted Euclidean distance (WED). Note that if \(\lambda \leq 0\), then, we get similar results as in the generalized mean. For example, if \(\lambda = 1\), we get the weighted harmonic distance and if \(\lambda = 0\), we obtain some kind of weighted geometric distance. The problem when using \(\lambda \leq 0\), is that sometimes the results are inconsistent. Especially, this problem appears in the weighted geometric distance because if one of the individual distances is 0, then, the whole aggregation is also 0.

2.2. INDUCED OWA OPERATOR

The IOWA operator was introduced by Yager and Filev [16] and it is an extension of the OWA operator. Its main difference is that the reordering step is not developed with the values of the arguments \(a_i\). In this case, the reordering step is developed with order inducing variables. It can be defined as follows.

Definition 3. An IOWA operator of dimension n is a mapping \(f: \mathbb{R}^n \rightarrow \mathbb{R}\) that has an associated weighting vector \(W\) of dimension n such that the sum of the weights is 1 and \(w_j \in [0,1]\), and its argument variable consists of \(n\)-tuple of pairs, then:

\[
f(\langle a_1, a_2, \ldots, a_n \rangle) = \sum_{j=1}^{n} w_j b_j
\]

where \(b_j\) is the \(j\)th largest \(a_i\) of the IOWA pair \(\langle a_i, a_j \rangle\) having the \(j\)th largest \(a_i\), \(a_j\) is the order inducing variable and \(a_i\) is the argument variable.

The IOWA operator includes the OWA operator as a particular case and a lot of other situations such as the maximum, the minimum and the average. Note that it is possible to distinguish between the Descending IOWA (DIOWA) operator and the Ascending IOWA (AIOWA) operator.

3. THE INDUCED MINKOWSKI ORDERED WEIGHTED AVERAGING DISTANCE OPERATOR

The IMOWAD operator is a distance measure that uses the IOWA operator in the normalization process of the Minkowski distance. Then, the reordering of the individual distances is developed with order inducing variables. For two sets X and Y, it can be defined as follows.

Definition 4. An IMOWAD operator is a mapping \(f: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}\) that has an associated weighting vector \(W\) such that \(w_j \in [0,1]\) and the sum of the weights is 1, then:

\[
f(\langle x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \rangle) = \left( \frac{\sum_{j=1}^{n} w_j b_j^\lambda}{\sum_{j=1}^{n} w_j b_j^\lambda} \right)^{1/\lambda}
\]
where $b_i$ is the $|y_i - y_j|$ value of the IMOWAD triplet $(u_i, x_i, y_i)$ having the $j$th largest $u_i$. $u_i$ is the order inducing variable and $|y_i - y_j|$ is the argument variable represented in the form of individual distances and $\lambda$ is a parameter such that $\lambda \in (-\infty, \infty)$.

**Remark 1:** The IMOWAD operator is commutative, monotonic, bounded and idempotent. It is commutative because any permutation of the arguments has the same evaluation. That is, $f((u_1, x_1, y_1), (u_2, x_2, y_2), \ldots, (u_n, x_n, y_n)) = f((u_1, x_1, y_1), (u_2, x_2, y_2), \ldots, (u_n, x_n, y_n))$ where $((u_1, x_1, y_1), (u_2, x_2, y_2), \ldots, (u_n, x_n, y_n))$ is any permutation of the arguments $((u_1, x_1, y_1), (u_2, x_2, y_2), \ldots, (u_n, x_n, y_n))$. It is monotonic because if $|x_i - y| \geq |x_j - y_j|$, for all $i$, then, $f((u_1, x_1, y_1), (u_2, x_2, y_2), \ldots, (u_n, x_n, y_n))$ is bounded because the IMOWAD aggregation is delimited by the minimum and the maximum. That is, $\text{Min}(|x_i - y|) \leq f((u_1, x_1, y_1), (u_2, x_2, y_2), \ldots, (u_n, x_n, y_n)) \leq \text{Max}(|x_i - y|)$. It is idempotent because if $|x_i - y| = |x_j - y|$, for all $i$, then, $f((u_1, x_1, y_1), \ldots, (u_n, x_n, y_n)) = (u_n, x_n, y_n)$.

**Remark 2:** Note that if $x_i = y_i$ for all $i \in [1, n]$, $f((u_1, x_1, y_1), \ldots, (u_n, x_n, y_n)) = 0$. Note also that $f((u_1, x_1, y_1), \ldots, (u_n, x_n, y_n)) = f((u_1, x_1, y_1), \ldots, (u_n, x_n, y_n))$.

**Remark 3:** From a generalized perspective of the reordering step it is possible to distinguish between descending (DIMOWAD) and ascending (AIMOWAD) orders. The weights of these operators are related by $w_j = w_{n+1-j}$, where $w_j$ is the $j$th weight of the DIMOWAD (or IMOWAD) operator and $w_{n+1-j}$ the $j$th weight of the AIMOWAD operator.

**Remark 4:** If $B$ is a vector corresponding to the ordered arguments $b_i$, we shall call this the ordered argument vector and $W^T$ is the transpose of the weighting vector, then, the IMOWAD operator can be expressed as:

$$f((u_1, x_1, y_1), \ldots, (u_n, x_n, y_n)) = (W^T B)^{1/\lambda}$$

(5)

**Remark 5:** Note that if the weighting vector is not normalized, i.e., $W = \sum_{j=1}^n w_j \neq 1$, then, the IMOWAD operator can be expressed as:

$$f((u_1, x_1, y_1), \ldots, (u_n, x_n, y_n)) = \left(\frac{1}{W} \sum_{j=1}^n w_j b_i^2\right)^{1/\lambda}$$

(6)

**Remark 6:** A further interesting issue is the problem of ties in the reordering process of the order inducing variables. In order to solve this problem, we recommend to follow the policy explained in [16] about replacing the tied arguments by their average. Note that in this case, it would mean that we are replacing the tied arguments by their normalized Minkowski distance.

### 4 Families of IMOWAD Operators

In this Section, we analyze different particular cases of the IMOWAD operator. We distinguish between those families found in the parameter $\lambda$ and those found in the weighting vector $W$.

#### 4.1. Analysing the Parameter $\lambda$

By looking at the parameter $\lambda$, we can find a wide range of distance measures such as the IOWAD, the EIOWAD, the induced ordered weighted geometric distance (IOWGD) operator, the induced ordered weighted harmonic averaging distance (IOWHD) operator and a lot of other cases. Note that in the IMOWAD we also find similar properties than the WMD when $\lambda \leq 0$. Therefore, we may find some inconsistencies, especially in the IOWGD operator.

**Remark 7:** When $\lambda = 1$, the IMOWAD operator becomes the IOWAD operator.

$$f((u_1, x_1, y_1), \ldots, (u_n, x_n, y_n)) = \sum_{j=1}^n w_j b_j$$

(7)

Note that if $w_j = 1/n$, for all $a$, we get the NHD. The WHD is obtained if $u_i > u_{i+1}$, for all $i$, and the OAWD operator is obtained if the ordered position of $u_i$ is the same than the ordered position of $b_i$ such that $b_i$ is the $j$th largest of $|x_i - y_i|$. Note also that it is possible to distinguish between descending (DIOWAD) and ascending (AIOWAD) orders.

**Remark 8:** When $\lambda = 2$, we get the IEOWAD operator.

$$f((u_1, x_1, y_1), \ldots, (u_n, x_n, y_n)) = \left(\sum_{j=1}^n w_j b_j^2\right)^{1/2}$$

(8)

If $w_j = 1/n$, for all $a$, we get the NED. If $u_i > u_{i+1}$, for all $i$, we get the WED and if the ordered position of $u_i$ is the same than the ordered position of $b_i$ such that $b_i$ is the $j$th largest of $|x_i - y_i|^2$, we get the EOWD operator. In this case, we also get the descending IEOWAD (DIEOWAD) operator and the ascending IEOWAD (AIEOWAD) operator.

**Remark 9:** When $\lambda = 0$, we get the IOWGD operator.
Remark 10: When \( \lambda = -1 \), we get the IOWHAD operator.

\[
f((u_1, x_1, y_1), \ldots, (u_n, x_n, y_n)) = \prod_{j=1}^{n} b_j^{w_j}
\]

Note that if \( w_j = 1/n \), for all \( a_i \), we get the normalized geometric distance and if \( u_i > u_{i+1} \), for all \( i \), the weighted geometric distance. If the ordered position of \( u_i \) is the same than the ordered position of \( b_j \) such that \( b_j \) is the \( j \)th largest of \( |x_i - y_i| \), we get the ordered weighted geometric distance operator (OWGD) operator. In this case, it is also possible to distinguish between descending (DIOWGD) and ascending (AIOWGD) orders. Note that the IOWGD can only be used sometimes when all the individual distances are different from 0. That is, when \( |x_i - y_i| \neq 0 \), for all \( i \).

Remark 11: Note that we could analyze other families by using different values in the parameter \( \lambda \). Also note that it is possible to study these families individually in a similar way as it has been developed in Section 3.

4.2. ANALYSING THE WEIGHTING VECTOR W

By choosing a different manifestation of the weighting vector in the IMOWAD operator, we are able to obtain different types of distance aggregation operators. For example, we can obtain the maximum distance, the minimum distance, the NMD, the WMD and the MOWAD operator.

The maximum distance is obtained if \( w_j = 1 \) and \( w_j = 0 \), for all \( j \neq p \), and \( u_p = \max(a_p) \). The minimum distance is obtained if \( w_j = 1 \) and \( w_j = 0 \), for all \( j \neq p \), and \( u_p = \min(a_p) \). More generally, if \( w_j = 1 \) and \( w_j = 0 \), for all \( j \neq k \), we are using the step-IMOWAD operator. The NMD is found when \( w_j = 1/m \), for all \( a_i \). The WMD is obtained if \( u_j > u_{j+1} \), for all \( i \), and the MOWAD operator is obtained if the ordered position of \( u_i \) is the same than the ordered position of \( b_j \) such that \( b_j \) is the \( j \)th largest of \( a_i \).

Remark 12: Other families of IMOWAD operators could be used. For more information on the methodology of these families, see for example [6,10,13]. For example, when \( w_j = 1/m \) for \( k \leq j \leq k + m - 1 \) and \( w_j = 0 \) for \( j > k + m \) and \( j < k \), we are using the window-IMOWAD operator. Note that \( k \) and \( m \) must be positive integers such that \( k + m - 1 \leq n \).

Remark 13: If \( w_j = w_\alpha = 0 \), and for all others \( w_j = 1/(n - 2) \), we are using the olympic-IMOWAD. Note that if \( n = 3 \) or \( n = 4 \), the olympic-IMOWAD is transformed in the IMOWAD median and if \( m = n - 2 \) and \( k = 2 \), the window-IMOWAD is transformed in the olympic-IMOWAD. Also note that the olympic-IMOWAD is transformed in the olympic-MOWAD average if \( w_p = w_q = 0 \), such that \( u_p = \max\{a_i\} \) and \( u_q = \min\{a_i\} \), and for all others \( w_j = 1/(n - 2) \).

Remark 14: Note that the IMOWAD-median and the weighted IMOWAD-median can also be used as a particular case of the IMOWAD. For the IMOWAD median, if \( n \) is odd we assign \( w_{(n+1)/2} = 1 \) and \( w_j = 0 \) for all others, and this affects the argument \( a_i \) with \( \{n + 1/2\} \)th largest \( u_k \). If \( n \) is even we assign for example, \( w_{n/2} = w_{(n/2) + 1} = 0.5 \), and this affects the arguments with \( \{n/2\}\)th and \( \{(n/2) + 1\}\)th largest \( u_k \). For the weighted IMOWAD median, we select the argument \( a_i \) that has the \( k \)th largest inducing variable \( u_i \), such that the sum of the weights from 1 to \( k \) is equal or higher than 0.5 and the sum of the weights from 1 to \( k - 1 \) is less than 0.5. Note that if the ordered position of \( u_i \) is the same than the ordered position of \( b_j \) such that \( b_j \) is the \( j \)th largest of \( a_i \), then, we get the MOWAD-median and the weighted MOWAD-median, respectively.

Remark 15: Another interesting family is the S-IMOWAD operator. It can be divided in three classes, the “orlike”, the “andlike” and the generalized S-IMOWAD operator. The generalized S-IMOWAD operator is obtained when \( w_j = (1/n)(1 - (\alpha + \beta)) + \alpha \) with \( u_p = \max\{a_i\} \); \( w_j = (1/n)(1 - (\alpha + \beta)) + \beta \), with \( u_q = \min\{a_i\} \); and \( w_j = (1/n)(1 - (\alpha + \beta)) \) for all \( j \neq p, q \) where \( \alpha, \beta \in [0, 1] \) and \( \alpha + \beta \leq 1 \). Note that if \( \alpha = 0 \), the generalized S-IMOWAD operator becomes the “andlike” S-IMOWAD operator and if \( \beta = 0 \), it becomes the “orlike” S-IMOWAD operator.

Remark 16: A further interesting family that could be used is the centered-IMOWAD operator. An IMOWAD operator is a centered aggregation operator if it is symmetric, strongly decaying and inclusive. It is symmetric if \( w_j = w_{n-j+1} \). It is strongly decaying when \( i < j \leq (n + 1)/2 \) then \( w_i < w_j \) and when \( i > j \geq (n + 1)/2 \) then \( w_i > w_j \). It is inclusive if \( w_0 > 0 \). Note that it is possible to consider a softening of the second condition by using \( w_i \leq w_j \) instead of \( w_i < w_j \). We shall refer to this as softly decaying-centered-IMOWAD operator. Another particular
situation of the centered-IMOWAD operator appears if we remove the third condition. We will refer to it as a non-inclusive centered-IMOWAD operator.

5 QUASI-IOWAD OPERATOR

The IMOWAD can be generalized by using quasi-arithmetic means in a similar way as it was done in [1-4,6]. We will call it the Quasi-IOWAD operator. It is defined as follows.

**Definition 5.** A Quasi-IOWAD operator is a mapping \( f: R^n \times R^m \rightarrow R \) that has an associated weighting vector \( W \) such that \( w_j \in [0, 1] \) and the sum of the weights is 1, then:

\[
f(\langle u_1, x_1, y_1 \rangle, \ldots, \langle u_n, x_n, y_n \rangle) = \frac{1}{n} \sum_{j=1}^{n} w_j g(b_j)
\]

where \( b_j = |x_j - y_j| \) is the \( j \)th largest \( x_j, u_j \), the order inducing variable, \( |x_j - y_j| \) is the argument variable represented in the form of individual distances, and \( g \) is the strictly continuous monotonic function.

As we can see, when \( g(b) = b^4 \), then, the Quasi-IOWAD becomes the IMOWAD operator. Note that it is also possible to distinguish between descending (Quasi-DIOWAD) and ascending (Quasi-AIOWAD) orders.

**Remark 17:** Note that all the properties and particular cases commented in the IMOWAD operator are also applicable in the Quasi-IOWAD operator such as the distinction between descending and ascending orders, different families of Quasi-IOWAD operators, the normalization process of the weighting vector when it is different from 1 by using \( (1 / W) \), etc.

6 NUMERICAL EXAMPLE

In the following, we are going to develop an illustrative example in order to see the results obtained in the aggregation by using different types of IMOWAD operators. We will analyze the selection of investments where an enterprise is looking for the best strategy according to its interests.

Assume that an enterprise that operates in Europe and North America wants to invest some money the next year. In order to do this, the board of directors has established four possible investments \( S_i \), that the enterprise could develop in the future.

1. \( A_1 \): Invest in the Asian market.
2. \( A_2 \): Invest in the South American market.
3. \( A_3 \): Invest in the African market.
4. \( A_4 \): Do not develop any investment.

After careful review of the information, the experts have given the following general information. They have summarized the information of the strategies in five main characteristics \( C_i \) with the following results.

1. \( C_1 \): Risk of the investment.
2. \( C_2 \): Benefits in the short term.
3. \( C_3 \): Benefits in the long term.
4. \( C_4 \): Difficulty of the investment.
5. \( C_5 \): Other aspects.

Note that the results are valuations between 0 and 1.

**Table 1: Available investments**

<table>
<thead>
<tr>
<th>( C_1 )</th>
<th>( C_2 )</th>
<th>( C_3 )</th>
<th>( C_4 )</th>
<th>( C_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_1 )</td>
<td>0.7</td>
<td>0.9</td>
<td>0.8</td>
<td>0.7</td>
</tr>
<tr>
<td>( A_2 )</td>
<td>0.6</td>
<td>0.8</td>
<td>0.7</td>
<td>0.5</td>
</tr>
<tr>
<td>( A_3 )</td>
<td>0.5</td>
<td>0.6</td>
<td>0.8</td>
<td>0.4</td>
</tr>
<tr>
<td>( A_4 )</td>
<td>0.8</td>
<td>0.5</td>
<td>0.6</td>
<td>0.8</td>
</tr>
</tbody>
</table>

According to the objectives and policies of the enterprise, the experts have established the ideal investment for the company independently of the investments available. They have established the following valuations for it.

**Table 2: Characteristics of the ideal investment**

<table>
<thead>
<tr>
<th>( C_1 )</th>
<th>( C_2 )</th>
<th>( C_3 )</th>
<th>( C_4 )</th>
<th>( C_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I )</td>
<td>0.8</td>
<td>0.9</td>
<td>1</td>
<td>0.9</td>
</tr>
</tbody>
</table>

In order to aggregate the information, the group of experts calculates the attitudinal character of the enterprise. Due to the fact that the attitudinal character depends upon the opinion of several members of the board of directors, it is very complex. Therefore, they to use order inducing variables in the reordering process. The results are shown in Table 3.

**Table 3: Order inducing variables**

<table>
<thead>
<tr>
<th>( C_1 )</th>
<th>( C_2 )</th>
<th>( C_3 )</th>
<th>( C_4 )</th>
<th>( C_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_1 )</td>
<td>12</td>
<td>16</td>
<td>20</td>
<td>24</td>
</tr>
<tr>
<td>( A_2 )</td>
<td>22</td>
<td>18</td>
<td>20</td>
<td>26</td>
</tr>
<tr>
<td>( A_3 )</td>
<td>14</td>
<td>20</td>
<td>15</td>
<td>18</td>
</tr>
<tr>
<td>( A_4 )</td>
<td>26</td>
<td>21</td>
<td>19</td>
<td>15</td>
</tr>
</tbody>
</table>

With this information, it is possible to develop different methods for selecting an investment. In this example, we will consider the NHD, the NED, the WHD, the WED, the OWAD, the IOWAD, the AIOWAD and the EIOWAD operator. Note that the weighting vector used
is: $W = (0.1, 0.2, 0.2, 0.2, 0.3)$. The aggregated results are shown in Tables 4 and 5.

<table>
<thead>
<tr>
<th></th>
<th>NHD</th>
<th>WHD</th>
<th>OWAD</th>
<th>IOWAD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>0.22</td>
<td>0.27</td>
<td>0.16</td>
<td>0.26</td>
</tr>
<tr>
<td>$A_2$</td>
<td>0.22</td>
<td>0.21</td>
<td>0.19</td>
<td>0.22</td>
</tr>
<tr>
<td>$A_3$</td>
<td>0.26</td>
<td>0.23</td>
<td>0.21</td>
<td>0.26</td>
</tr>
<tr>
<td>$A_4$</td>
<td>0.24</td>
<td>0.27</td>
<td>0.2</td>
<td>0.27</td>
</tr>
</tbody>
</table>

Table 5: Aggregated results 2

<table>
<thead>
<tr>
<th></th>
<th>AIOWAD</th>
<th>EIOWAD</th>
<th>Median</th>
<th>Olympic</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>0.18</td>
<td>0.349</td>
<td>0.2</td>
<td>0.1</td>
</tr>
<tr>
<td>$A_2$</td>
<td>0.22</td>
<td>0.249</td>
<td>0.2</td>
<td>0.3</td>
</tr>
<tr>
<td>$A_3$</td>
<td>0.26</td>
<td>0.306</td>
<td>0.3</td>
<td>0.233</td>
</tr>
<tr>
<td>$A_4$</td>
<td>0.21</td>
<td>0.305</td>
<td>0.3</td>
<td>0.3</td>
</tr>
</tbody>
</table>

As we can see, depending on the distance aggregation operator used, the optimal choice is different. Note that the lowest value in each method is the optimal result.

If we establish an ordering of the investments, a typical situation if we want to consider more than one alternative, we will get the following orders shown in Table 6. Note that the first alternative in each ordering is the optimal choice.

<table>
<thead>
<tr>
<th></th>
<th>Ordering</th>
</tr>
</thead>
<tbody>
<tr>
<td>NHD</td>
<td>$A_1$ $A_4$ $A_1$ $A_2$</td>
</tr>
<tr>
<td>WHD</td>
<td>$A_1$ $A_4$ $A_1$ $A_2$</td>
</tr>
<tr>
<td>OWAD</td>
<td>$A_1$ $A_4$ $A_1$ $A_2$</td>
</tr>
<tr>
<td>IOWAD</td>
<td>$A_1$ $A_4$ $A_1$ $A_2$</td>
</tr>
<tr>
<td>AIOWAD</td>
<td>$A_1$ $A_4$ $A_1$ $A_2$</td>
</tr>
<tr>
<td>EIOWAD</td>
<td>$A_1$ $A_4$ $A_1$ $A_2$</td>
</tr>
<tr>
<td>Median-IOWAD</td>
<td>$A_1$ $A_4$ $A_1$ $A_2$</td>
</tr>
<tr>
<td>Olympic-IOWAD</td>
<td>$A_1$ $A_4$ $A_1$ $A_2$</td>
</tr>
</tbody>
</table>

As we can see, depending on the particular type of IMOWAD operator used, the results may lead to different decisions.

7 CONCLUSIONS

We have presented the IMOWAD operator. It is a distance measure that uses the IOWA operator in the Minkowski distance. The main advantage of this operator is that it generalizes a wide range of distances such as the NMD, the WMD, the MOWAD, the IOWAD, the EIOWAD, etc. We have studied some of its main properties.

We have further generalized the IMOWAD operator by using quasi-arithmetic means. We have called it the Quasi-IOWAD. We have also developed an application of the new approach in a decision making problem about selection of investments. We have seen that the main advantage of using the IMOWAD is that it gives a more complete view of the decision problem.

In future research, we expect to develop further extensions of the IMOWAD operator by adding new characteristics in the problem and applying it to other fields.

Acknowledgements

We would like to thank the anonymous referees for their insightful comments that have led to an improved version of the paper.

References


